The Local Time of the Classical Risk Process*

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Abstract. In this paper we give an explicit expression for the local time of the classical risk process and associate it with the density of an occupational measure. To do so, we approximate the local time by a suitable sequence of absolutely continuous random fields. Also, as an application, we analyze the mean of the times $s \in [0,T]$ such that $0 \le X_s \le X_{s+\varepsilon}$ for some given $\varepsilon > 0$.

1. Introduction and main results

Henceforth, $X = \{X_t, t \ge 0\}$ represents the classical risk process. More precisely,

$$X_t = x_0 + ct - \sum_{k=1}^{N_t} R_k, \quad t \ge 0,$$

where $x_0 \ge 0$ is the initial capital, $c > 0$ is the premium income per unit of time, $N = \{N_t, t \ge 0\}$ is an homogeneous Poisson process with rate α and $\{R_k, k \in \mathbb{N}\}$ is a sequence of i.i.d non-negative random variables, which is independent of N. N_t is interpreted as the number of claims arrivals during time t and R_k as the amount of the k-th claim. We suppose that R_1 has finite mean and it is an absolutely continuous random variable with respect to the Lebesgue measure.

The risk process has been studied extensively because it is often used to describe the capital of an insurance company. Indeed, among the properties of X considered by several authors, we can metion that the local time of X has been analyzed by Kolkovska et al. [**7**], the double Laplace transform of an occupation measure of X has been obtained by Chiu and Yin [**3**], or that the probability of ruin has been one of the most important goals of

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the risk theory (see, for example, Asmussen [**1**], Grandell [**8**], Rolski et al. [**11**] and the references therein to get an idea of the analysis realized in this subject). In this paper we are interested in continuing the development of the local time L of X and its applications as an occupational density in order to improve the understanding of X.

Note that X is a Lévy process due to $\sum_{k=1}^{N} R_k$ being a compound Poisson process. Thus, we can apply different criteria for general Lévy processes to guarantee the existence of L. For example, we can use the Hawkes' result [**9**] when R_k is exponential distributed (see also [**2**] and references therein for related works). However, we cannot obtain in general the form of L via this results. Moreover, in the literature there exist different characterizations of the local time (see Fitzsimmons and Port [**6**] and the references therein). For instance, the local time have been introduced in [**6**] (resp. [**7**]) as an $L^2(\Omega)$-derivative (resp. derivative in probability) of some occupation measure. Nevertheless, in [**6, 7**], it is not analyzed some properties of the involved local time using this "approximation of L".

The purpose of this paper is to associate the local time of X with the crossing process when L is interpreted as a density of the occupational measure (see Theorem 1.c) below). The relation between the local time and the crossing process was conjectured by Lévy [**10**] for the Brownian motion case (i.e., whe X is a Wiener process). In this article we use the ideas of the proof of Tanaka's formula for the Brownian motion (see Chung [**4**], Chapter 7) to obtain a sequence of absolutely continuous random fields (in time) that converges with probability 1 (w.p.1 for short) to

$$\begin{aligned} L_t(x) &= \frac{1}{c}\Big(\frac{1}{2}1_{\{x\}}(X_t) + 1_{(x,\infty)}(X_t) - \frac{1}{2}1_{\{x\}}(x_0) - 1_{(x,\infty)}(x_0) \\ &\quad - \sum_{0<s\le t}\{1_{(x,\infty)}(X_s) - 1_{(x,\infty)}(X_{s-})\}\Big), \quad t\ge 0 \text{ and } x\in\mathbb{R}. \end{aligned} \tag{1.1}$$

This approximation allows us to prove that this L is the density of the occupation measure (see (1.3) below) and, therefore, to deal with some problems related to occupations measures.

Notice that L given by (1.1) is well-defined because X is càdlàg and

$$P(N_t < +\infty, \text{ for all } t>0) = 1, \tag{1.2}$$

wich imply that only a finite number of summands in (1.1) are different than zero.

In the following result we not only relate L to the number of crossings with certain level, but also to the occupation measure

$$Y_t(A) = \int_0^t 1_A(X_s)ds, \quad t\ge 0 \text{ and } A\in\mathcal{B}(\mathbb{R}), \tag{1.3}$$

where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algrebra of $\mathbb{R}$. Toward this end, we need the following:

DEFINITION 1. *We say that there exists a crossing with the level* $x \in \mathbb{R}$ *at time* $s \in (0, +\infty)$ *if for all open interval* I *such that* $s \in I$*,* x *is an interior point of* $\{X_t : t \in I\}$*. That is,* $x \in (X(I))^{\circ}$*. Moreover, the number of crossings with the level* x *in the interval* $(0, t)$ *is denoted by* $C_t(x)$*.* C *is known as the crossing process of* X*.*

Observe that if $x \in \mathbb{R}$ is a crossing point at time s, then X is continuous at time s and $X_s = x$.

Now we can state the main result of the paper.

THEOREM 1. *Let* $t > 0$ *and* $x \in \mathbb{R}$*. Then, the random field* L *defined in (1.1) has the following properties:*

a) $L_t(x) \geq 0$ *and* $L_{\cdot}(x)$ *is not decreasing w.p.1.*
b) $L_t(x) = \frac{1}{c}\left(\frac{1}{2}1_{\{X_t\}}(x) - \frac{1}{2}1_{\{X_0\}}(x) + C_t(x)\right)$ *w.p.1.*
c) *For every bounded and Borel measurable function* $g : \mathbb{R} \to \mathbb{R}$*, we have*

$$\int_0^t g(X_s)ds = \int_{\mathbb{R}} g(y)L_t(y)dy \quad w.p.1. \tag{1.4}$$

Note that Statement b) implies that the number of crossings C of X introduced in Definition 1 satisfies

$$C_t(x) = 1_{(-\infty, X_t)}(x) - 1_{(-\infty, X_0)}(x) + \sum_{0<s\leq t} 1_{(X_s, X_{s-})}(x) \quad w.p.1,$$

for $t > 0$ and $x \in \mathbb{R}$. Also note that, from (1.4) and Statement a), the random field L can be interpreted as an occupation density relative to the Lebesgue measure on $\mathbb{R}$. Hence, L in (1.1) is called *the local time* and the expression

$$\begin{aligned} L_t(x) \;\; &= \;\; \frac{1}{c}(\frac{1}{2}1_{\{X_t\}}(x) - \frac{1}{2}1_{\{X_0\}}(x) + 1_{(-\infty, X_t)}(x) - 1_{(-\infty, X_0)}(x) \\ &\quad + \int_{(0,t]} f(x, X_s)dX_s), \end{aligned}$$

is known as *Tanaka-like formula for* $L_t(x)$. Here

$$f(x, X_s) = \begin{cases} \frac{1_{(X_s, X_{s-})}(x)}{\Delta X_s}, & \Delta X_s \neq 0, \\ 0, & \Delta X_s = 0. \end{cases}$$

On the other hand, relation (1.4) can be extended to some occupational results. Indeed, as an example, we can state the following, which leads us to get some average of the pathwise behavior of X.

THEOREM 2. *Let* $g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a bounded and Borel measurable function. Then for each* $\varepsilon > 0$*,*

$$E[\int_0^t g(X_s, X_{s+\varepsilon} - X_s)ds] = \int_{\mathbb{R}} E[g(x, X_\varepsilon - x_0)]E[L_t(x)]dx. \tag{1.5}$$

An application of this theorem is to answer the question: *What is the average in time that the capital of an insurance company is positive, and bigger than itself after twelve months?.*

The paper is organized as follows. In Section 2 we provide the tool needed to prove Theorem 1. In particular, we approximate the local time by a sequence of suitable random fields. The proof of Theorem 1 is given in Section 3. Finally, in Section 4, we show Theorem 2 and answer the above question in the case that the claim R_1 has exponential distribution.

2. Main tool

In this section we provide the needed tool to show that Theorem 1 holds. In particular, we construct the announced sequence converging to the local time L.

In the remaining of this paper, T_i denotes the i-th jump time of N, with $T_0 = 0$. It is known that T_i has gamma distribution with parameters (i, α), $i \geq 1$.

We will use the following technical resul in the proofs of this section.

Lemma 1. *Let $x \in \mathbb{R}$, $s > 0$, $\Omega_1(s) = \{\Delta X_s \neq 0\}$, and*

$$\begin{aligned} \Omega_2 &= \{X_{s-} = x,\ \Delta X_s \neq 0 \textit{ for some } s > 0\} \\ &\quad \cup \{X_s = x,\ \Delta X_s \neq 0 \textit{ for some } s > 0\}. \end{aligned}$$

Then, $P(\Omega_1(s)) = 0$ and $P(\Omega_2) = 0$.

Proof. By the law of total probability

$$P(\Omega_1(s)) = \sum_{k=0}^{\infty} P(N_s = k) P(\Omega_1(s) | N_s = k).$$

Notice that

$$P(\Omega_1(s) | N_s = k) = P(\Delta X_s \neq 0 | N_s = k) = P(T_k = s) = 0.$$

On the other hand, let $\nu \in \mathbb{N}$ and define

$$\begin{aligned} \tilde{\Omega}_\nu &= \{X_{s-} = x,\ \Delta X_s \neq 0 \text{ for some } 0 < s < \nu\} \\ &\quad \cup \{X_s = x,\ \Delta X_s \neq 0 \text{ for some } 0 < s < \nu\}. \end{aligned}$$

For $k = 0$,

$$P(\tilde{\Omega}_\nu | N_\nu = 0) = P(\emptyset | N_\nu = 0) = 0,$$

and for $k \geq 1$,

$$\begin{aligned} P(\tilde{\Omega}_\nu | N_\nu = k) &\leq P(X_{T_j-} = x \text{ for some } j \in \{1, ..., k\} \, | N_\nu = k) \\ &\quad + P(X_{T_j} = x \text{ for some } j \in \{1, ..., k\} \, | N_\nu = k) \\ &\leq \sum_{j=1}^{k} (P(X_{T_j-} = x | N_\nu = k) + P(X_{T_j} = x | N_\nu = k)). \end{aligned}$$

For $j = 1$ we get

$$\begin{aligned} P(X_{T_1-} = x | N_\nu = k) &= P(T_1 = (x - x_0)c^{-1} | N_\nu = k) = 0, \\ P(X_{T_1} = x | N_\nu = k) &= P(cT_1 - R_1 = x - x_0 | N_\nu = k) = 0, \end{aligned}$$

this is because T_1 and R_1 are independent and absolutely continuous random variables. Let $P(\cdot | N_\nu = k) = P^*(\cdot)$. When $j > 1$ we have

$$\begin{aligned} P^*(X_{T_j-} = x) &= \int_{\mathbb{R}} P^*(X_{T_j-} = x | X_{T_{j-1}-} = y) P^*(X_{T_{j-1}-} \in dy) \\ &= \int_{\mathbb{R}} P^*(R_{j-1} = y - (x - (T_j - T_{j-1})c)) P^*(X_{T_{j-1}-} \in dy) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} P^*(X_{T_j} = x) &= \int_{\mathbb{R}} P^*(X_{T_j} = x | X_{T_{j-1}} = y) P^*(X_{T_{j-1}} \in dy) \\ &= \int_{\mathbb{R}} P^*((T_j - T_{j-1})c - R_j = x - y) P^*(X_{T_{j-1}} \in dy) \\ &= 0. \end{aligned}$$

Here we have used the fact that R_{j-1} has an absolutely continuous distribution. Finally notice that $P(\Omega_2) \leq \sum_{\nu=1}^{\infty} P(\tilde{\Omega}_\nu) = 0$. $\square$

2.1. An approximating sequence of the local time. Now we approximate the local time L by a sequence of suitable random fields, which allows us to see that Theorem 1.a) is true. Toward this end, let $x \in \mathbb{R}$ arbitrary and fixed. For each $n \in \mathbb{N}$ define $\varphi_{x,n} : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_{x,n}(y) = \begin{cases} 0, & y < x - 1/n, \\ (n(y-x)+1)/2, & x - 1/n \leq y \leq x + 1/n, \\ 1, & x + 1/n < y. \end{cases}$$

Notice that

$$\begin{aligned} \lim_{n\to\infty} \varphi_{x,n}(y) &= \begin{cases} 0, & y < x, \\ 1/2, & y = x, \\ 1, & y > x, \end{cases} \\ &= \frac{1}{2} 1_{\{x\}}(y) + 1_{(x,+\infty)}(y), \end{aligned} \tag{2.1}$$

and

$$\varphi'_{x,n}(y) = \begin{cases} 0, & y < x - 1/n, \\ n/2, & x - 1/n < y < x + 1/n, \\ 0, & x + 1/n < y. \end{cases}$$

For each $n \in \mathbb{N}$, we define the random field

$$\begin{aligned} L^n_t(x) &= \frac{1}{c}\left(\varphi_{x,n}(X_t) - \varphi_{x,n}(X_0)\right. \\ &\quad - \sum_{s\leq t}\{\varphi_{x,n}(X_s) - \varphi_{x,n}(X_{s-}) - \varphi'_{x,n}(X_{s-})\Delta X_s\}), \end{aligned}$$

where $\Delta X_t = X_t - X_{t-}$. As in (1.1), we have by (1.2) that L^n is well-defined.

Before proving that $\{L^n, n \in \mathbb{N}\}$ is the sequence that we are looking for, we need to approximate the fuction $\varphi_{x,n}$ by a sequence of smooth functions. To do so, set

$$\begin{aligned} \Omega' &= (\{X_{s-} = x \pm 1/m \neq X_s, \text{ for some } s > 0,\ m \in \mathbb{N}\} \\ &\quad \cup \{N_s < +\infty, \quad \text{for all} \quad s > 0\}^c \cup \Omega_2)^c. \end{aligned}$$

Since, by Lemma 1,

$$\begin{aligned} P(X_{s-} &= x \pm 1/m \neq X_s, \text{ for some } s > 0,\ m \in \mathbb{N}) \\ &\leq \sum_{m=1}^{\infty} P(X_{s-} = x \pm 1/m \neq X_s, \text{ for some } s > 0) = 0, \end{aligned}$$

we have $P(\Omega') = 1$.

Let $\psi : \mathbb{R} \to \mathbb{R}$ a symmetric function in $\mathcal{C}^\infty(\mathbb{R})$ with compact support on $[-1, 1]$ and

$$\int_{-1}^{1} \psi(y)dy = 1.$$

Define the sequence (ψ_m) by

$$\psi_m(y) = m\psi(my), \quad y \in \mathbb{R},$$

and

$$\varphi^m_{x,n}(y) = (\psi_m * \varphi_{x,n})(y) = \int_{\mathbb{R}} \varphi_{x,n}(y - z)\psi_m(z)dz.$$

Since $\psi_m \in \mathcal{C}^\infty(\mathbb{R})$, then $\varphi^m_{x,n} \in \mathcal{C}^\infty(\mathbb{R})$ and moreover

$$\left(\varphi^m_{x,n}\right)_m \text{ converges uniformly on compacts to } \varphi_{x,n}, \tag{2.2}$$

$$((\varphi^m_{x,n})')_m \text{ converges pointwise, except on } x \pm 1/n, \text{ to } (\varphi_{x,n})'. \tag{2.3}$$

Let us use the notation

$$L^{n,m}_t(x) = \frac{1}{c}\int_{(0,t]} (\varphi^m_{x,n})'(X_{s-})dX_s.$$

Then, by the change of variable theorem for the Lebesgue-Stieltjes integral, we have

$$\begin{aligned} cL_t^{n,m}(x) &= \varphi_{x,n}^m(X_t) - \varphi_{x,n}^m(x_0) \\ &\quad - \sum_{0<s\le t} \left\{ \varphi_{x,n}^m(X_s) - \varphi_{x,n}^m(X_{s-}) - (\varphi_{x,n}^m)'(X_{s-})\Delta X_s \right\}. \end{aligned} \tag{2.4}$$

Now we can give the relation between L^n and $\{L^{n,m}, m \in \mathbb{N}\}$.

PROPOSITION 1. *Let $n \in \mathbb{N}$. Then,*

$$\lim_{m\to\infty} L_t^{n,m}(x) = L_t^n(x), \quad \textit{for all } t > 0, \ \textit{w.p.1}.$$

PROOF. For $\omega \in \Omega'$ we have that (1.2) and (2.2) imply

$$\begin{aligned} &\lim_{m\to\infty} (\varphi_{x,n}^m(X_t) - \varphi_{x,n}^m(x_0) - \sum_{0<s\le t} \left\{ \varphi_{x,n}^m(X_s) - \varphi_{x,n}^m(X_{s-}) \right\}) \\ &\qquad = \varphi_{x,n}(X_t) - \varphi_{x,n}(x_0) - \sum_{0<s\le t} \{\varphi_{x,n}(X_s) - \varphi_{x,n}(X_{s-})\}. \end{aligned}$$

Now we analyze the remaining term in the definition of $L_t^{n,m}(x)$. Notice that for each $w \in \Omega'$ we have

$$X_{s-}(w) \neq x \pm 1/k, \quad k \in \mathbb{N}.$$

Therefore, from (2.3),

$$\lim_{m\to\infty} \sum_{0<s\le t} (\varphi_{x,n}^m)'(X_{s-})\Delta X_s = \sum_{0<s\le t} (\varphi_{x,n})'(X_{s-})\Delta X_s.$$

From this and (2.4) the result follows. □

Now we are ready to state the properties of $\{L^n, n \in \mathbb{N}\}$ that we use in Section 3.

PROPOSITION 2. *The sequence $\{L^n, n \in \mathbb{N}\}$ fulfill:*

a) $L_t^n(x) = \int_0^t \varphi'_{x,n}(X_{s-})ds + \frac{1}{c}\sum_{0<s\le t} \varphi'_{x,n}(X_{s-})\Delta X_s$, *for all $n \in \mathbb{N}$ and $t > 0$, w.p.1.*
b) $\lim_{n\to\infty} L_t^n(x) = L_t(x)$, *for all $t > 0$, w.p.1.*

PROOF. We first deal with Statement a). Fix $t \geq 0$ and let $\omega \in \Omega' \cap \Omega_1(t)^c$. Then there is $k \in \mathbb{N}$ such that $N_t(\omega) = k$. Thus

$$\begin{aligned}
&\int_0^t (\varphi_{x,n}^m)'(X_{s-})dX_s \\
&= \sum_{i=1}^k \int_{(T_{i-1},T_i]} (\varphi_{x,n}^m)'(X_{s-})dX_s + \int_{(T_k,t]} (\varphi_{x,n}^m)'(X_{s-})dX_s \\
&= \sum_{i=1}^k \int_{(T_{i-1},T_i]} \left(\varphi_{x,n}^m\right)'(X_{s-})cds + \int_{[T_k,t)} \left(\varphi_{x,n}^m\right)'(X_s)cds \\
&\quad + \sum_{i=1}^k \left(\varphi_{x,n}^m\right)'(X_{T_i-})(X_{T_i} - X_{T_i-}) \\
&= c\int_0^t (\varphi_{x,n}^m)'(X_{s-})ds + \sum_{i=1}^k \left(\varphi_{x,n}^m\right)'(X_{T_i-})(X_{T_i} - X_{T_i-}).
\end{aligned}$$

Notice that on each $(T_{i-1}, T_i]$ and $(T_k, t]$, there is at most one s such that $X_{s-} = x \pm 1/n$. Hence, by (2.3), we have

$$\lim_{m\to\infty} (\varphi_{x,n}^m)'(X_{s-}) = (\varphi_{x,n})'(X_{s-}), \quad \lambda\text{-}a.s.$$

Therefore, by the dominated convergence theorem, we deduce

$$\begin{aligned}
\lim_{m\to\infty} \int_0^t (\varphi_{x,n}^m)'(X_{s-})dX_s \quad &= \quad c\int_0^t (\varphi_{x,n})'(X_{s-})ds \\
&\quad + \sum_{i=1}^k \left(\varphi_{x,n}\right)'(X_{T_i-})\Delta X_{T_i}, \quad a.s.
\end{aligned}$$

Consequently, the fact that $L_t^n(x)$ and the right-hand side of last equality are càdlàg processes implies that Statement a) holds.

Now we consider Statement b) in order to finish the proof of the proposition. Let $\omega \in \Omega'$ and $t \geq 0$. Then there exist $k \in \mathbb{N}$ such that $N_t(\omega) = k$. Hence

$$\begin{aligned}
&\lim_{n\to\infty} (\varphi_{x,n}(X_t(\omega)) - \varphi_{x,n}(x_0) - \sum_{i=1}^k \{\varphi_{x,n}(X_{T_i}(\omega)) - \varphi_{x,n}(X_{T_i-}(\omega))\}) \\
&= \frac{1}{2}1_{\{x\}}(X_t(\omega)) + 1_{(x,\infty)}(X_t(\omega)) - \frac{1}{2}1_{\{x\}}(x_0) - 1_{(x,\infty)}(x_0) \\
&\quad - \sum_{i=1}^k \{\frac{1}{2}1_{\{x\}}(X_{T_i}(\omega)) + 1_{(x,\infty)}(X_{T_i}(\omega)) \\
&\qquad - \frac{1}{2}1_{\{x\}}(X_{T_i-}(\omega)) - 1_{(x,\infty)}(X_{T_i-}(\omega))\}.
\end{aligned}$$

On the other hand

$$\omega \in \{X_{s-} = x \neq X_s, \text{ for some } s > 0\}^c$$

implies

$$X_{T_i-}(w) \neq x, \quad i = 0, 1, ..., k.$$

Here there exist a finite number of indexes i such that

$$X_{T_i}(w) \leq x < X_{T_i-}(w).$$

For large enough n we have

$$X_{T_i-}(\omega) \notin (x - 1/n, x + 1/n), \quad i = 0, 1, ..., k.$$

Therefore

$$\begin{aligned} &\lim_{n\to\infty} \sum_{0<s\leq t} (\varphi_{x,n})'(X_{s-}(\omega))\Delta X_s(\omega) \\ &= \lim_{n\to\infty} \sum_{i=1}^{k} (\varphi_{x,n})'(X_{T_i-}(\omega))\Delta X_{T_i}(\omega) \\ &= \lim_{n\to\infty} \sum_{i=1}^{k} \frac{n}{2} 1_{(x-1/n,x+1/n)}(X_{T_i-}(\omega))\Delta X_{T_i}(\omega) = 0. \end{aligned}$$

Hence the proof is complete. □

3. Proof of Theorem 1

The purpose of this section is to give the proof of Theorem 1. This proof will be divided into three steps. It is worth mentioning that the proof of Statement a) gives us a sequence of absolutely continuous random fields that converges to L. Namely, the sequence $\{\int_0^t \varphi'_{x,n}(X_{s-})\, ds, n \in \mathbb{N}\}$.

3.1. Proof of part a) of Theorem 1. From part a) and b) of Proposition 2 we have

$$L_t(x) = \lim_{n\to\infty} L_t^n(x) = \lim_{n\to\infty} \int_0^t \varphi'_{x,n}(X_{s-})\, ds.$$

which yields that $(L_\cdot^n(x))$ is non-negative and increasing.

3.2. Proof of part b) of Theorem 1. Since $X_s \leq X_{s-}$ we have

$$\begin{aligned} L_t(x) \;&=\; \frac{1}{c}\Big(\frac{1}{2}1_{\{X_t\}}(x) - \frac{1}{2}1_{\{x_0\}}(x) + 1_{(-\infty,X_t)}(x) - 1_{(-\infty,x_0)}(x) \\ &\quad + \sum_{0<s\leq t} 1_{(X_s,X_{s-})}(x)\Big). \end{aligned} \tag{3.1}$$

Suppose for example, $x_0 > x$, $X_t < x$ and $C_t(x) = n$. Let $c_1, ..., c_n$ the crossing times with the level x. Then, by hypothesis, there exist jumping times $s_1 \in (0, c_1)$,...,$s_{n+1} \in (c_n, t)$ such that $x \in (X_{s_i}, X_{s_i-})$. Hence

$$\begin{aligned} 1_{(-\infty, X_t)}(x) - 1_{(-\infty, x_0]}(x) + \sum_{0<s\leq t} 1_{(X_s, X_{s-})}(x) &= 0 - 1 + (n+1) \\ &= C_t(x). \end{aligned}$$

3.3. Proof of part c) of Theorem 1. For each $a, b \in \mathbb{R}$ define

$$\begin{aligned} 1_{\langle\langle a,b\rangle\rangle} &= \begin{cases} 1_{(a,b]}, & \text{if } a \leq b, \\ -1_{(b,a]}, & \text{if } b < a, \end{cases} \\ &= 1_{(-\infty,b]} - 1_{(-\infty,a]}. \end{aligned}$$

From this definition immediately follows that

$$1_{\langle\langle a,b\rangle\rangle} = 1_{(a,c]} - 1_{(b,c]}, \quad a, b \leq c. \tag{3.2}$$

Using induction on n, we can prove that, for $a_1, ..., a_n$ real numbers,

$$1_{\langle\langle a_1,a_2\rangle\rangle} + \cdots + 1_{\langle\langle a_{n-1},a_n\rangle\rangle} = 1_{\langle\langle a_1,a_n\rangle\rangle}. \tag{3.3}$$

On the other hand, for almost all $\omega \in \Omega'$, there exists $k \in \mathbb{N} \cup \{0\}$ such that $\omega \in \{N_t = k\}$. Therefore,

$$\begin{aligned} \int_0^t g(X_s)ds &= \sum_{i=1}^k \int_{(T_{i-1},T_i]} g(X_s)ds + \int_{(T_k,t]} g(X_s)ds \\ &= \sum_{i=1}^k \int_{(T_{i-1},T_i]} g(X_{T_{i-1}} + c(s - T_{i-1}))ds \\ &\quad + \int_{(T_k,t]} g(X_{T_k} + c(s - T_k))ds. \end{aligned}$$

Taking $x = X_{T_{i-1}} + (s - T_{i-1})c$, we can write

$$\begin{aligned} \int_0^t g(X_s)ds &= \sum_{i=1}^k \int_{(X_{T_{i-1}}, X_{T_{i-1}} + c(T_i - T_{i-1})]} g(x)\frac{dx}{c} \\ &\quad + \int_{(X_{T_k}, X_{T_k} + c(t-T_k)]} g(x)\frac{dx}{c} \\ &= \sum_{i=1}^k \int_{(X_{T_{i-1}}, X_{T_i-}]} g(x)\frac{dx}{c} + \int_{(X_{T_k}, X_t]} g(x)\frac{dx}{c} \\ &= \frac{1}{c}\int_{\mathbb{R}} g(x) \sum_{i=1}^k 1_{(X_{T_{i-1}}, X_{T_i-}]}(x)dx \\ &\quad + \frac{1}{c}\int_{\mathbb{R}} g(x) 1_{(X_{T_k}, X_t]}(x)dx. \end{aligned}$$

From (3.2) and (3.3) we have

$$\begin{aligned}
\int_0^t g(X_s)ds &= \frac{1}{c}\int_{\mathbb{R}} g(x)\sum_{i=1}^{k} 1_{(X_{T_i},X_{T_i-}]}(x)dx \\
&\quad +\frac{1}{c}\int_{\mathbb{R}} g(x)\sum_{i=1}^{k} 1_{\langle\langle X_{T_{i-1}},X_{T_i}\rangle\rangle}(x)dx \\
&\quad +\frac{1}{c}\int_{\mathbb{R}} g(x)1_{(X_{T_k},X_t]}(x)dx \\
&= \frac{1}{c}\int_{\mathbb{R}} g(x)\sum_{i=1}^{k} 1_{(X_{T_i},X_{T_i-}]}(x)dx \\
&\quad +\frac{1}{c}\int_{\mathbb{R}} g(x)1_{\langle\langle X_{T_0},X_{T_k}\rangle\rangle}(x)dx \\
&\quad +\frac{1}{c}\int_{\mathbb{R}} g(x)1_{(X_{T_k},X_t]}(x)dx \\
&= \frac{1}{c}\int_{\mathbb{R}} (1_{\langle\langle X_0,X_t\rangle\rangle}+\sum_{i=1}^{k} 1_{(X_{T_i},X_{T_i-}]})(x)g(x)dx \\
&= \int_{\mathbb{R}} \frac{1}{c}(1_{(-\infty,X_t]}-1_{(-\infty,X_0]}+\sum_{i=1}^{k} 1_{(X_{T_i},X_{T_i-}]})(x)g(x)dx.
\end{aligned}$$

Thus, the proof is complete by (3.1).

4. An occupation measure result

By $F(\cdot,t)$ we denote the distribution of $\sum_{k=1}^{N_t} R_k 1_{[N_t>0]}$, and by $f(\cdot,t)$ the density of $F(\cdot,t)$, when it exists. In order to use Theorem 2 we need an expression for $E[L_t(x)]$, which is given in [**7**] (Proposition 1). Namely, if $f \in L^1(\mathbb{R}\times[0,t])$, then

$$E[L_t(x)] = \int_{[((x-x_0)/c)\vee 0]\wedge t}^{t} f(x_0+cs-x,s)ds. \tag{4.1}$$

4.1. Example. Consider the measurable set

$$\Delta = [0,\infty)\times[0,\infty)\in\mathcal{B}(\mathbb{R}^2).$$

Then, from Theorem 2 and (4.1), we get

$$\begin{aligned}
&E[\int_0^t 1_\Delta(X_s, X_{s+\varepsilon}-X_s)ds] \\
&= \int_{\mathbb{R}} E[1_\Delta(x, X_\varepsilon - x_0)] \int_{[((x-x_0)/c)\vee 0]\wedge t}^t f(x_0+cs-x,s)dsdx \\
&= \int_0^\infty P(x_0 \le X_\varepsilon) \int_{[((x-x_0)/c)\vee 0]\wedge t}^t f(x_0+cs-x,s)dsdx \\
&= \int_0^\infty P(\sum_{k=1}^{N_\varepsilon} R_k \le c\varepsilon) \int_{[((x-x_0)/c)\vee 0]\wedge t}^t f(x_0+cs-x,s)dsdx \\
&= \int_0^\infty F(c\varepsilon,\varepsilon) \int_{[((x-x_0)/c)\vee 0]\wedge t}^t f(x_0+cs-x,s)dsdx.
\end{aligned}$$

Now assume that R_1 has exponential distribution with parameter β, then the density of $\sum_{k=1}^{N_t} R_k 1_{[N_t>0]}$ is

$$f(x,t) = e^{-\alpha t-\beta x}\left(\sum_{n=1}^\infty \frac{(\beta\alpha t)^n x^{n-1}}{n!(n-1)!}\right) 1_{(0,\infty)}(x), \quad t>0.$$

Hence, in this case,

$$\begin{aligned}
&E[\int_0^t 1_\Delta(X_s, X_{s+\varepsilon}-X_s)ds] \\
&= \int_0^\infty \left[\int_0^{c\varepsilon} e^{-\alpha\varepsilon-\beta y}\sum_{n=1}^\infty \frac{(\beta\alpha\varepsilon)^n y^{n-1}}{n!(n-1)!}dy + e^{-\alpha\varepsilon}\right] \\
&\times \int_{[((x-x_0)/c)\vee 0]\wedge t}^t e^{-\alpha s}e^{-\beta(x_0+cs-x)}\sum_{k=1}^\infty \frac{(\beta\alpha s)^k (x_0+cs-x)^{k-1}}{k!(k-1)!}dsdx \\
&= \int_0^{x_0} \left[\int_0^{c\varepsilon} e^{-\alpha\varepsilon-\beta y}\sum_{n=1}^\infty \frac{(\beta\alpha\varepsilon)^n y^{n-1}}{n!(n-1)!}dy + e^{-\alpha\varepsilon}\right] \\
&\times \int_0^t e^{-\alpha s}e^{-\beta(x_0+cs-x)}\sum_{k=1}^\infty \frac{(\beta\alpha s)^k (x_0+cs-x)^{k-1}}{k!(k-1)!}dsdx \\
&+ \int_{x_0}^{x_0+ct} \left[\int_0^{c\varepsilon} e^{-\alpha\varepsilon-\beta y}\sum_{n=1}^\infty \frac{(\beta\alpha\varepsilon)^n y^{n-1}}{n!(n-1)!}dy + e^{-\alpha\varepsilon}\right] \\
&\times \int_{(x-x_0)/c}^t e^{-\alpha s}e^{-\beta(x_0+cs-x)}\sum_{k=1}^\infty \frac{(\beta\alpha s)^k (x_0+cs-x)^{k-1}}{k!(k-1)!}dsdx.
\end{aligned}$$

For example, under the conditions

$$x_0 = 4,\ \alpha = 1,\ \beta = 1,\ c = 1.1,\ t = 1,$$

with $\varepsilon = 12$ and considering five iterations on the sums we get

$$
\begin{aligned}
&E[\int_0^1 1_\Delta(X_s, X_{s+12} - X_s)ds] \\
&\approx \int_0^4 \left[\int_0^{13.2} e^{-12-y} \sum_{n=1}^{5} \frac{(12)^n y^{n-1}}{n!(n-1)!} dy + e^{-12}\right] \\
&\times \int_0^1 e^{-(2.1)s-4+x} \sum_{k=1}^{5} \frac{s^k(4+(1.1)s-x)^{k-1}}{k!(k-1)!} dsdx \\
&+ \int_4^{5.1} \left[\int_0^{13.2} e^{-12-y} \sum_{n=1}^{5} \frac{(12)^n y^{n-1}}{n!(n-1)!} dy + e^{-12}\right] \\
&\times \int_{(x-4)/(1.1)}^1 e^{-(2.1)s-4+x} \sum_{k=1}^{5} \frac{s^k(4+(1.1)s-x)^{k-1}}{k!(k-1)!} dsdx \\
&= 7.251 \times 10^{-3}.
\end{aligned}
$$

Note that this value may help the insurance company to decide if it invests part of its wealth in another assets.

4.2. Proof of Theorem 2. We will use the monotone class theorem (see, for example, Ethier and Kurtz [**5**], Theorem 4.2) to show that the result holds. Set

$$\mathcal{H} = \{\psi : \mathbb{R}^2 \longrightarrow \mathbb{R},\ \psi \text{ is measurable, bounded and satisfies (1.5)}\}.$$

It is not difficult to see that $\mathcal{H}$ is a real linear space and, by Theorem 1, we have

$$\int_{\mathbb{R}} E[L_t(x)]dx = E[\int_{\mathbb{R}} L_t(x)dx] = E[\int_0^t 1_{\mathbb{R}}(X_s)ds] = t.$$

It means, $1_{\mathbb{R}^2} \in \mathcal{H}$. Moreover $\mathcal{H}$ is closed under monotone convergence: Let $(\psi_n) \subset \mathcal{H}$, such that $0 \le \psi_n \uparrow \psi$, ψ bounded, then ψ is measurable and

$$
\begin{aligned}
E[\int_0^t \psi(X_s, X_{s+\varepsilon} - X_s)ds] &= \lim_{n\to\infty} E[\int_0^t \psi_n(X_s, X_{s+\varepsilon} - X_s)ds] \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} E[\psi_n(x, X_\varepsilon - x_0)]E[L_t(x)]dx \\
&= \int_{\mathbb{R}} E[\psi(x, X_\varepsilon - x_0)]E[L_t(x)]dx,
\end{aligned}
$$

which gives that $\psi \in \mathcal{H}$.

Now we use the notation

$$\mathcal{K} = \{\psi : \mathbb{R}^2 \longrightarrow \mathbb{R},\ \psi(\cdot, \cdot\cdot) = 1_A(\cdot)1_B(\cdot\cdot),\quad A, B \in \mathcal{B}(\mathbb{R})\}.$$

Then the family $\mathcal{K}$ is closed under multiplication and $\mathcal{K} \subset \mathcal{H}$. In fact, by Theorem 1 we obtain

$$\begin{aligned}
&E[\int_0^t 1_A(X_s)1_B(X_{s+\varepsilon}-X_s)ds] \\
&= \int_0^t E[1_A(X_s)]E[1_B(X_{s+\varepsilon}-X_s)]ds \\
&= \int_0^t E[1_A(X_s)]E[1_B(\varepsilon c - \sum_{k=N_s+1}^{N_{s+\varepsilon}} R_k)]ds \\
&= \int_0^t E[1_A(X_s)]E[1_B(\varepsilon c - \sum_{k=1}^{N_\varepsilon} R_k)]ds \\
&= \int_0^t E[1_A(X_s)]E[1_B(X_\varepsilon - x_0)]ds \\
&= E[1_B(X_\varepsilon - x_0)] \int_0^t E[1_A(X_s)]ds \\
&= E[1_B(X_\varepsilon - x_0)]E[\int_{\mathbb{R}} 1_A(x)L_t(x)dx] \\
&= \int_{\mathbb{R}} E[1_A(x)1_B(X_\varepsilon - x_0)]E[L_t(x)]dx.
\end{aligned}$$

Finally, the Dynkin monotone class theorem yields that the proof is finished.

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